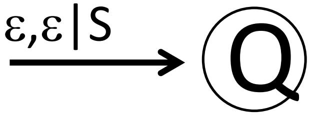
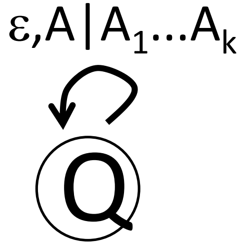
Equivalence of PDAs and Grammars

Theorem: Every language described by a context-free grammar is accepted by a PDA.
Construction: Start with a grammar for the language, where S is the start symbol. Make a start state Q for the DFA and begin

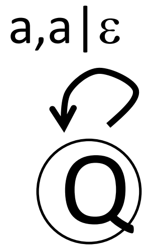


For each grammar rule A => $A_1..A_k$ add transition



i.e. push A_k, then A_{k-1}, etc., finally pushing A_1.

Construction continued: For each terminal symbol a in Σ add the transition



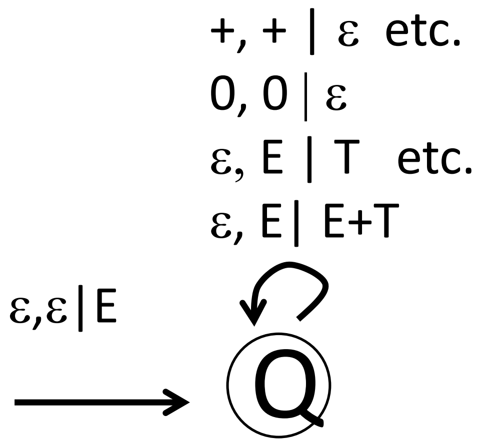
This completes the construction. Note that the DFA has only one state. It accepts by empty stack.

Example:

E => E+T | T

T => T*F | F

F => F digit | digit



Following is a configuration analysis that shows this DFA accepts 3+4*5

(Q, 3+4*5, E) => (Q, 3+4*5, E+T)

=> (Q,3+4*5, T+T)

=> (Q,3+4*5, F+T)

=> (Q,3+4*5, 3+T)

=> (Q,+4*5, +T)

=> (Q,4*5, T)

=> (Q,4*5, T*F)

=> (Q,4*5, F*F)

=> (Q,4*5, 4*F)

=> (Q,*5, *F)

=> (Q,5, F)

=> (Q,5,5)

=>(Q,ε,ε) accept

Now, how do we know this PDA accepts the language generated by the grammar?

Suppose string w is generated by the grammar. Then there is a derivation of w that always expands the left-most nonterminal symbol:

E => E+T
=> T+T
=> F+T
etc.

At each step i let A_i be the left-most nonterminal, α_i everything to its left, and β_i everything to its right so the phrase that has been derived is $\alpha_i A_i \beta_i$ and all of the symbols in α_i are terminal.

The automaton has been constructed so that at step i of the automaton computation the stack will be $A_i\beta_i$ and the α_i symbols of the input will have been consumed. In other words, an easy induction shows that

$$(Q,w,S) \overset{*}{\Rightarrow} (Q, w-\alpha_i, A_i\beta_i)$$

So eventually $(Q,w,S) \overset{*}{\Rightarrow} (Q, \varepsilon, \varepsilon)$ and the automaton accepts w.

On the other hand, suppose that for a nonterminal symbol A $(Q, w, A) \overset{*}{\Rightarrow} (Q,\varepsilon,\varepsilon)$. We will show by induction that there is a grammar derivation of w from symbol A. The induction is on the number of moves made by the automaton.

Base case: There must be a grammar rule A=>a and w=a.
Inductive case: Suppose this is true for all strings accepted by the PDA in n moves and the PDA accepts w in n+1 moves.
Since the configuration (Q, w, A) starts with a nonterminal at the top of the stack the first move must be using a rule $A \Rightarrow X_1..X_k$. For each i let w_i be the string of input needed to remove X_i from the stack, i.e.,

$$(Q,w_i,X_i) \overset{*}{\Rightarrow} (Q,\varepsilon,\varepsilon)$$

By induction $X_i \overset{*}{\Rightarrow} w_i$.

Altogether A => $X_1..X_k \overset{*}{\Rightarrow} w_1..w_k=w$. So if the automaton accepts w the grammar derives w.

Theorem (Chomsky): Given a PDA that accepts by empty stack, we can find a context free grammar that generates the set of strings accepted by the PDA.
Construction: This builds a huge grammar whose derivations mimic the configurations of the PDA.

Step 1. The nonterminal symbols of the grammar are a new start symbol S and all symbols of the form [pXq] where p and q are states of the PDA and X is any one stack symbol

[pXq] will generate all strings w so that $(p,w,X) \overset{*}{\Rightarrow} (q,\varepsilon,\varepsilon)$
i.e., all strings w that take the PDA from state p to state q while popping X off the stack.

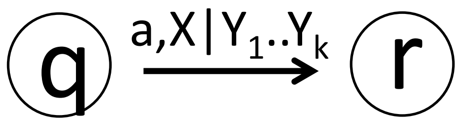
Step 2. Grammar rules

Rule 1: If Q is the start state of the PDA and Z_0 is the stack bottom symbol then for every state p add the grammar rule

$$S \Rightarrow [QZ_0p]$$

i.e., S will generate all strings that take the PDA from Q to any other state while emptying the stack.

Rule 2: Suppose the PDA has transition



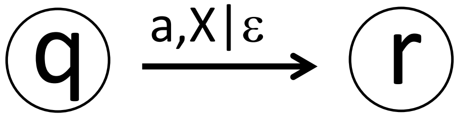
Then for every sequence of k states $r_1..r_k$ add the rule

$[qXr_k] \Rightarrow a[rY_1r_1][r_1Y_2r_2] \ldots [r_{k-1}Y_kr_k]$

i.e., the strings that take the PDA from q to r_k while removing X from the stack include those that

1. first consume a and move from q to r
2. then consume anything generated by $[rY_1r_1]$
3. then consume anything generated by $[r_1Y_2r_2]$
4. etc.

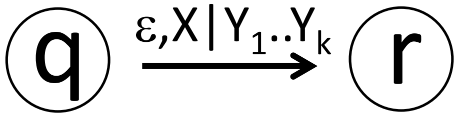
Rule 3: If there is a transition



then add the rule

$$[qXr] \Rightarrow a$$

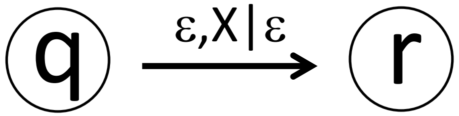
Rule 4: If there is a transition



then for any sequence of states $r_1..r_k$ add the rule

$$[qXr_k] \Rightarrow [rY_1r_1][r_1Y_2r_2] \ldots [r_{k-1}Y_kr_k]$$

Rule 5: If there is a transition

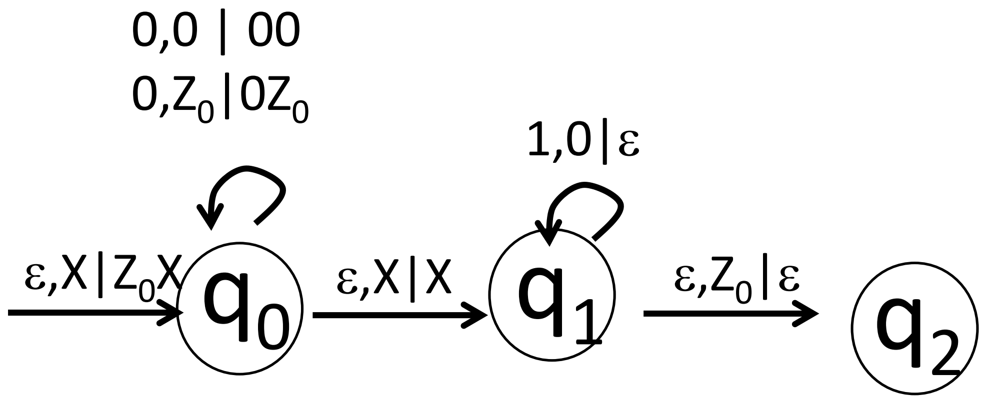


then add the rule

$$[qXr] \Rightarrow \varepsilon$$

This is the complete construction.

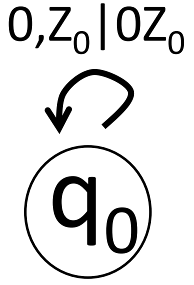
Example: The following automaton accepts $\{0^n1^n \mid n >= 0\}$ by empty stack



Here is a derivation of 0011 with the constructed grammar:

$S \Rightarrow [q_0Z_0q_2]$ Rule 1 with $p=q_2$ since Z_0 is popped at q_2.

$\Rightarrow 0[q_0 0 q_1][q_1Z_0q_2]$ Rule 2 with $q,r=q_0$, $r_1=q_1$, $r_2=q_2$

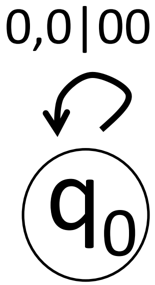


$\Rightarrow 00\underline{[q_0 0 q_1]}[q_1 0 q_1][q_1 Z_0 q_2]$

Rule 2 with

$q, r = q_0$,

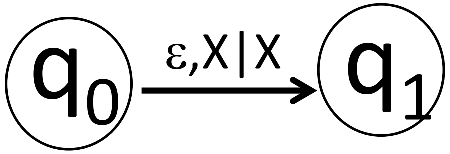
$r_1 = r_2 = q_1$



$\Rightarrow 00\underline{[q_1 0 q_1][q_1 0 q_1]}[q_1 Z_0 q_2]$

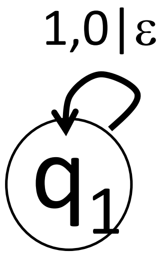
Rule 4 with

$r = q_1 = r_1$



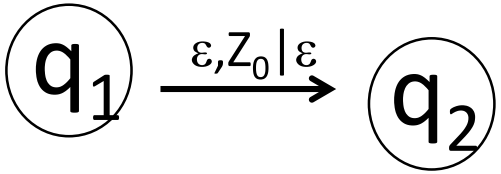
$=> 0011[q_1 Z_0 q_2]$

Rule 3 twice with

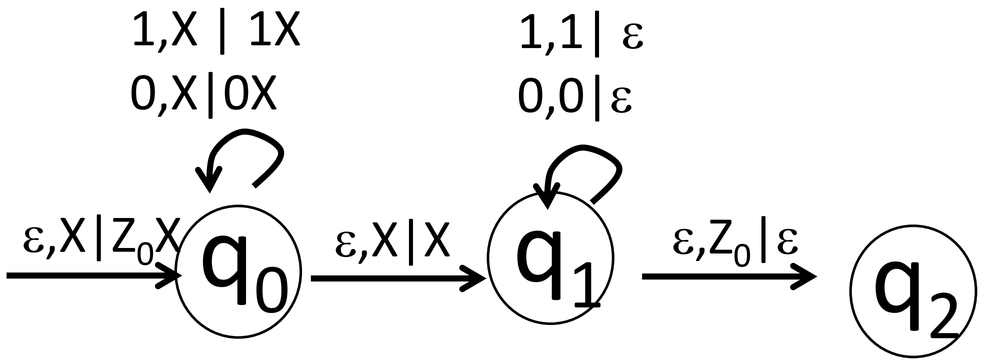


$=> 0011$

Rule 5 with



Another example



This accepts by empty stack $\{ww^{rev} \mid w \in (0+1)^* \}$

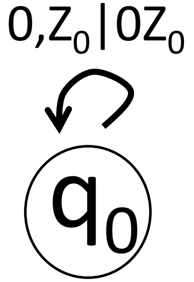
We will derive 0110 from the constructed grammar.

S => $[q_0 Z_0 q_2]$ Rule 1

=> $0[q_0 0 q_1][q_1 Z_0 q_2]$

Rule 2 with

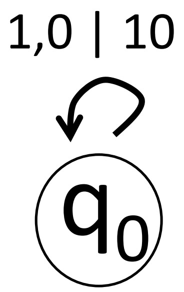
$r=q_0$, $r_1=q_1$, $r_2=q_2$



$\Rightarrow 01\underline{[q_0 1 q_1]}[q_1 0 q_1][q_1 Z_0 q_2]$

Rule 2 with

$r=q_0,\ r_1=q_1,\ r_2=q_1$



$\Rightarrow 01\underline{[q_1 1 q_1][q_1 0 q_1]}[q_1 Z_0 q_2]$

Rule 4 with

r=q1, r1=q1

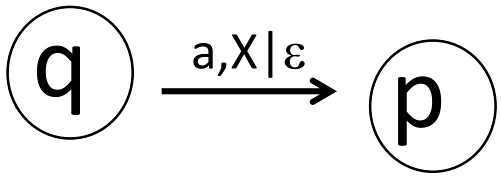
$\Rightarrow 0110\underline{[q_1 Z_0 q_2]}$ Rule 3 twice

$\Rightarrow 0110$ Rule 5

Lemma 1: If string w can take the PDA from state q to state p while popping X off the stack then $[qXp] \overset{*}{\Rightarrow} w$. As a consequence, if w is accepted by the PDA it is generated by the grammar.

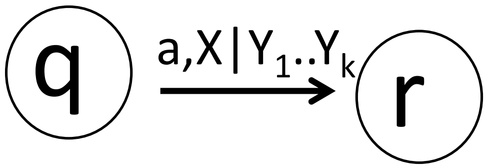
Proof of Lemma 1: Induction on the number of steps the PDA takes to transform configuration (q,w,X) to $(p,\varepsilon,\varepsilon)$

Base case: 1 step. The step must be (q,w,X) => $(p,\varepsilon,\varepsilon)$ so the PDA must have a transition



This means the grammar has a rule [qXp] => a (Rule 3)

Inductive case: Suppose the lemma is true for all strings w that take n or fewer steps in the configuration computation, and w takes n+1 steps. The first step must use a transition of the form



By Rule 2 the grammar will have a rule of the form
(*) $[qXp]\text{=>}a[rY_1r_1][r_1Y_2r_2]...[r_{k-1}Y_kp]$ for any sequence (r_i) of states.

Let w_i be the input that pops Y_i off the stack; let r_i be the state where this is completed.

By the inductive hypothesis we must have (**) $[r_{i-1}Y_ir_i]\overset{*}{\Rightarrow}w_i$

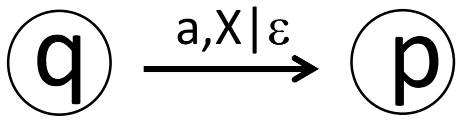
Putting (*) and (**) together we have

$$[qXp] \overset{*}{\Rightarrow} aw_1w_2...w_k=w$$

Lemma 2: If $[qXp]\overset{*}{\Rightarrow}w$ then $(q,w,X) \overset{*}{\Rightarrow} (p,\varepsilon,\varepsilon)$. As a consequence, if a string is generated by the grammar it is accepted by the PDA.

Proof of Lemma 2: We do induction on the number of steps in the grammar derivation $[qXp]\overset{*}{\Rightarrow}w$.

Base case: 1 step. There must be a rule [qXp]=>a, so it must come from a transition

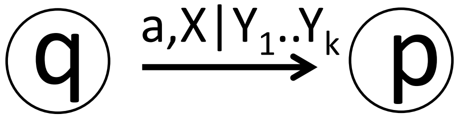


So $(q,a,X) \overset{*}{\Rightarrow} (p,\varepsilon,\varepsilon)$

Inductive case: Suppose this is true of all derivations of n or fewer steps and we have one with n+1 steps.

The first step must have the form $[qXp] \Rightarrow a[ry_1r_1][r_1Y_2r_2]...[r_{k-1}Y_kp]$

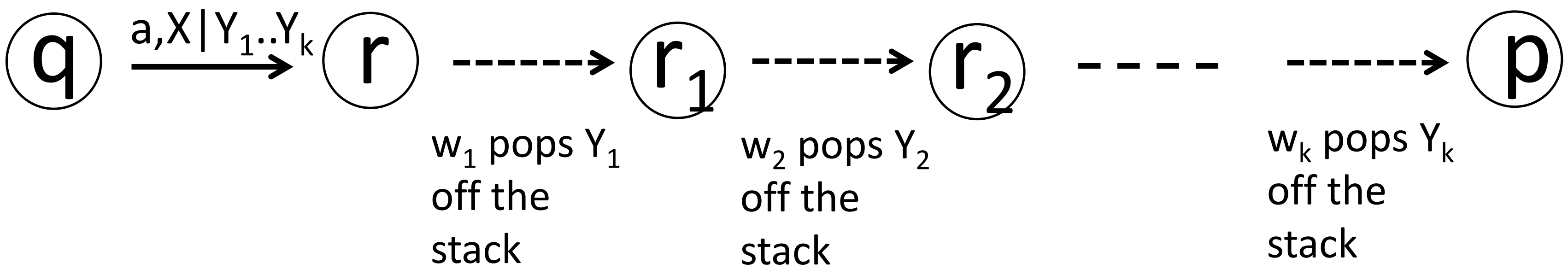
For this to be a grammar rule the PDA must have a transition



Each $[r_{i-1}Y_ir_i]$ symbol must generate a string of terminal symbols; call this string w_i.

By induction $(r_{i-1}, w_i, y_i) \overset{*}{\Rightarrow} (r_i, \varepsilon, \varepsilon)$

In other words the automaton goes through a series of transitions:



i.e., $aw_1w_2..w_k$ takes the automaton from q to p while popping X off the stack.